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Research Commentary

Reconceptualizing Procedural Knowledge

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In this article, I argue for a renewed focus in mathematics education research on procedural knowledge. I make three main points: (1) The development of students' procedural knowledge has not received a great deal of attention in recent research; (2) one possible explanation for this deficiency is that current characterizations of conceptual and procedural knowledge reflect limiting assumptions about how procedures are known; and (3) reconceptualizing procedural knowledge to remedy these assumptions would have important implications for both research and practice.

Key words: Algorithm; Conceptual knowledge; Flexibility; Heuristic; Procedural knowledge

The respective roles of procedural and conceptual knowledge in students' learning of mathematics continue to be a topic of animated conversation in the mathematics education community. As a prominent mathematics educator (Sowder, 1998) noted several years ago, "Whether developing skills with symbols leads to conceptual understanding, or whether the presence of basic understanding should precede symbolic representation and skill practice, is one of the basic disagreements" between the opposing sides of the so-called math wars. Among those who argue against current reform efforts, there is a perception that procedural knowledge acquisition has been de-emphasized and deemed less important than conceptual knowledge, with dire consequences for student learning (e.g., Budd et al., 2005; Mathematically Correct, n.d.). Although some reformers might disagree with this characterization, others are quite explicit in their belief that procedural knowledge *should* play a secondary, supporting role to conceptual knowledge in students' learning of mathematics (e.g., Pesek & Kirschner, 2000). Some go so far as to state that an instructional focus on procedural knowledge, rather than conceptual knowledge, leads to the development of isolated skills and rote knowledge, and that "a rush for procedural skill will actually do more harm than good" (Brown, Seidelmann, & Zimmermann, n.d.).

This issue has deep roots in our field (e.g., Brownell, 1945; Hiebert & Lefevre, 1986; Skemp, 1976); the current math wars indicate that we still have not reached consensus on the respective roles of procedural knowledge and conceptual knowl-

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edge in student learning. In fact, there may be more (and more vitriolic) debate on this topic now than at any time in the recent past, particularly with respect to procedural skill acquisition. Why is that the case? In this article, I reflect upon the nature of the conversation about procedures and concepts by making three points. First, I claim that disagreements on the role of procedural knowledge in mathematics learning are primarily ideological rather than empirical: We have not devoted a great deal of attention in our research to procedural knowledge and its development. Although we want students to use procedures "flexibly, accurately, efficiently, and appropriately" (National Research Council, 2001, p. 116), we do not know a lot about what this instructional outcome looks like, much less how it might develop. Second, I claim that a reason for the relative lack of research on procedural knowledge is that current characterizations of the terms *procedural knowledge* and *conceptual knowledge* are limiting and are in fact impediments to careful investigations of these constructs. Third, I argue that reconceptualizing procedural knowledge—and making it a renewed focus of research—would have important implications for both research and practice.

LACK OF RECENT RESEARCH ON THE DEVELOPMENT OF PROCEDURAL KNOWLEDGE

A survey of journals and publicly available databases indicates that the development of students' procedural knowledge has not been a recent focus of research in mathematics education. A key word search of the Educational Resources Information Center (ERIC) database for *procedural fluency,* a term recently promoted by the National Research Council (2001) in *Adding It Up,* yielded no articles. Perhaps given the newness of the term, this void may not be surprising. ERIC also indicates, however, that the ratio of journal articles in mathematics education that use the terms *conceptual knowledge* or *conceptual understanding* to those that use the terms *procedural knowledge* or *procedural skill* is approximately 4:1. Similarly, an ERIC key word search of the past 10 years of the *Journal for Research in Mathematics Education (JRME)* for *procedure* or *algorithm* yielded six articles, only four of which were even peripherally related to students' knowledge of procedures. Perhaps most convincingly, of the approximately 100 empirical articles related to the development of K–12 students' mathematical content knowledge published over the past decade in the *JRME,* in only 11 did the researchers carefully investigate the development of students' knowledge of procedures. Although this survey is far from exhaustive, it suggests that the ways that students come to know, use, and understand mathematical procedures have not been a prominent focus of mathematics education research for at least 10 years.

Procedures were widely studied in the 1980s, when many studies focused on students' procedural errors (e.g., Brown & VanLehn, 1980; Matz, 1980). In addition, a large amount of literature on procedural skill acquisition exists in cognitive psychology (e.g., Anderson, Fincham, & Douglass, 1997), and the relationship between procedural and conceptual knowledge continues to be a topic of research in developmental psychology (e.g., Rittle-Johnson, Siegler, & Alibali, 2001). But for at least the past 10 years, mathematics education researchers have largely avoided detailed and careful studies of the development of procedural skill.

Why is that the case? It is perhaps no coincidence that the relative lack of research on procedures has occurred in a time of political strife for mathematics educators. As alluded to above, the development of procedural skill and its role in K–12 instruction have been particularly contentious issues in the math wars, which might explain some researchers' reluctance to pursue this topic. In addition, and political reasons notwithstanding, some researchers may believe that procedural knowledge should not be a focus of research or instruction, perhaps because of a perception that skills are no longer of sufficient instructional importance (compared with conceptual knowledge) to justify studies of interventions primarily designed to improve procedural knowledge. Other researchers may feel that the widespread availability of technological tools has reduced or eliminated the need to study pedagogical and cognitive issues associated with the learning of procedural skills. There is evidence, however, that many mathematics educators continue to believe that procedural skill plays a fundamental and vital role in students' learning of mathematics (Ballheim, 1999; National Research Council, 2001). Note that I am not claiming that procedural knowledge is more important than conceptual knowledge. Rather, I claim that both are critical components of students' mathematical proficiency and thus merit careful study.

I propose a complementary explanation for the lack of mathematics education research on the development of procedural knowledge—namely, that current characterizations of conceptual and procedural knowledge reflect implicit and largely unfounded assumptions about how concepts and procedures are known.

CURRENT CHARACTERIZATIONS OF CONCEPTUAL AND PROCEDURAL KNOWLEDGE

The widespread use of the terms *conceptual knowledge* and *procedural knowledge* can be attributed to the seminal book edited by Hiebert (1986), particularly the introductory chapter by Hiebert and Lefevre (1986). They define conceptual knowledge as

> knowledge that is rich in relationships. It can be thought of as a connected web of knowledge, a network in which the linking relationships are as prominent as the discrete pieces of information. Relationships pervade the individual facts and propositions so that all pieces of information are linked to some network. (pp. 3–4)

Procedural knowledge is defined as follows:

> One kind of procedural knowledge is a familiarity with the individual symbols of the system and with the syntactic conventions for acceptable configurations of symbols. The second kind of procedural knowledge consists of rules or procedures for solving mathematical problems. Many of the procedures that students possess probably are chains of prescriptions for manipulating symbols. (pp. 7–8)

A close look at these excerpts reveals that conceptual knowledge is not defined as knowledge of concepts or principles, as a parsing of the phrase might suggest. Rather, it is defined in terms of the *quality* of one's knowledge of concepts—particularly the richness of the connections inherent in such knowledge. This definition is a critical departure from psychological views of concepts, especially in its depiction of conceptual knowledge as rich in relationships. The term *concept* does imply connected knowledge, whether one is speaking of mathematical concepts (e.g., limit, slope) or concepts more broadly (e.g., furniture, bicycle). But psychologically speaking, knowledge of a concept is not necessarily rich in relationships (Medin, 1989): The connections inherent in a concept may be only limited and superficial, or they may be extensive and deep.[1] For example, a very young child's conceptual knowledge of *dog* may be less deep, sophisticated, and connected than an adult's (Gelman, Star, & Flukes, 2002); a similar point could be made about the difference between a 6th grader's and an 11th grader's conceptual knowledge of slope. The point is that mathematics educators who strictly adhere to Hiebert and Lefevre's (1986) definition implicitly refer only to a particular subset of conceptual knowledge: that which is richly connected or deep.

What about procedural knowledge? Hiebert and Lefevre (1986) define this term essentially as knowledge of procedures: knowledge of the syntax, steps, conventions, and rules for manipulating symbols. In terms of the quality of knowledge implicit in the definition, Hiebert and Lefevre (1986) suggest that the relationships present in procedural knowledge are primarily sequential: A step in a procedure is connected to the next step. By this definition, procedural knowledge is superficial; it is not rich in connections. As was the case above, this definition is a significant departure from psychological perspectives on procedures. There are many different kinds of procedures, and the quality of the connections within a procedure varies (Anderson, 1982). Some procedures are algorithms, meaning that if one executes the procedure's steps in a predetermined order and without error, one is guaranteed to reach the problem's solution. Algorithms are apparently what Hiebert and Lefevre had in mind when they crafted their definition of procedural knowledge; in algorithms, it is the case that sequential relationships predominate. But other procedures are heuristics, meaning rules of thumb or somewhat general and more abstract procedures that may be helpful in solving a problem. Heuristic procedures are tremendously powerful assets in problem solving (Schoenfeld, 1979). The execution of heuristics requires that one make choices; wise choices can indicate quite sophisticated and deep knowledge. Mathematics educators who strictly adhere to Hiebert and Lefevre's definition of procedural knowledge are referring only to knowledge of algorithms; for this subset of proce-

[1] Deep-level knowledge has been structured and stored in memory in a way that makes it maximally useful for the performance of tasks, whereas surface- or superficial-level knowledge is associated with rote learning, inflexibility, reproduction, and trial and error (Glaser, 1991). Deep-level knowledge is associated with comprehension, abstraction, flexibility, critical judgment, and evaluation (De Jong & Ferguson-Hessler, 1996).

dures, it is reasonable to depict algorithmic knowledge as typically superficial, fully compiled, or rote (Anderson, 1992). Heuristics, however, are procedures too, and the Hiebert and Lefevre definition does not account for them.[2]

Hiebert and Lefevre's (1986) definitions of procedural and conceptual knowledge were quite influential in providing mathematics educators with a well-defined terminology to refer to students' knowledge of mathematics. However, the preceding discussion illustrates that these terms suffer from a entanglement of knowledge *type* and knowledge *quality* (De Jong & Ferguson-Hessler, 1996; Star, 2000) that makes their use somewhat problematic, especially for procedural knowledge. The term *conceptual knowledge* has come to encompass not only what is known (knowledge of concepts) but also one way that concepts can be known (e.g., deeply and with rich connections). Similarly, the term *procedural knowledge* indicates not only what is known (knowledge of procedures) but also one way that procedures (algorithms) can be known (e.g., superficially and without rich connections).

If knowledge type and knowledge quality have become conflated, then what would it mean to disentangle them? Consider the 2 × 2 matrix shown in Table 1. The matrix suggests that for both knowledge types (knowledge of concepts and knowledge of procedures), one can have knowledge that is either superficial or deep. The current usage of the terms *conceptual knowledge* and *procedural knowledge* makes it difficult to consider (or even name) the knowledge that belongs in the deep procedural knowledge cell.[3] Deep procedural knowledge would be knowledge of procedures that is associated with comprehension, flexibility, and critical judgment and that is distinct from (but possibly related to) knowledge of concepts. Separating these independent characteristics of knowledge (type versus quality) allows for the reconceptualization of procedural knowledge as potentially deep.

Table 1
Types and Qualities of Procedural and Conceptual Knowledge

Knowledge type	Knowledge quality	
	Superficial	Deep
Procedural	Common usage of *procedural knowledge*	?
Conceptual	?	Common usage of *conceptual knowledge*

[2] Hiebert and Lefevre (1986) acknowledge that their definitions of procedural and conceptual knowledge do not account for heuristics. They write, "No sooner than we propose definitions for conceptual and procedural knowledge and attempt to clarify them, we must back up and acknowledge that the definitions we have given and the impressions they convey will be flawed in some way. As we have said, not all knowledge fits nicely into one class or the other. Some knowledge lies at the intersection. Heuristic strategies for solving problems, which themselves are objects of thought, are examples" (p. 9).

[3] The cell for superficial conceptual knowledge was alluded to in the preceding discussion of conceptual knowledge. Knowledge of concepts certainly involves relationships, but those relationships are not necessarily deep or rich. A learner's initial knowledge of a concept is typically quite superficial and fragile, but over time the relationships can deepen and become richer.

What does deep procedural knowledge look like? Inspiration for this enhanced view of procedural knowledge can be found in research from the 1980s and early 1990s (e.g., Davis, 1983; Ohlsson & Rees, 1991; VanLehn, 1990). For example, VanLehn proposed that a student can have *teleological* understanding of a procedure, meaning knowledge of its design or justification for its use. Similarly, Davis writes of knowledge of procedures that might include such things as the order of steps, the goals and subgoals of steps, the environment or type of situation in which the procedure is used, the constraints imposed upon the procedure by the environment or situation, and any heuristics or common sense knowledge that are inherent in the environment or situation. Both of these examples illustrate procedural knowledge that is rich in relationships.

My own work on the development of procedural flexibility provides a more concrete and recent example (Star, 2000, 2002a, 2001/2002b; Star & Seifert, in press). When students use formal methods to solve linear equations in algebra, they have available a very limited set of actions: adding to or subtracting from both sides, combining like terms, distributing or factoring, and multiplying or dividing both sides. Yet despite that limitation, there is a wide array of problem types. Skilled equation solvers have the ability to use the equation-solving actions flexibly, so that a maximally efficient solution can be generated for any problem type. I consider flexibility to be an indicator of deep procedural knowledge.

Flexibility is a nontrivial and often overlooked competency. Consider three relatively simple (and superficially quite similar) linear equations: (a) $2(x + 1) + 3(x + 1) = 10$; (b) $2(x + 1) + 3(x + 1) = 11$; and (c) $2(x + 1) + 3(x + 2) = 10$. Although each of these equations can be solved with the same sequence of steps (using a standard algorithm for solving linear equations), the most efficient strategy may not be the standard algorithm. Furthermore, what is meant by the most efficient strategy is quite nuanced. Is the most efficient strategy the one that is the quickest or easiest to do, the one with the fewest steps, the one that avoids the use of fractions, or the one that the solver likes the best? There are subtle interactions among the problem's characteristics, one's knowledge of procedures, and one's problem-solving goals that might lead a solver to implement a particular series of procedural actions. Someone with only superficial knowledge of procedures likely has no recourse but to use a standard technique, which may lead to less efficient solutions or even an inability to solve unfamiliar problems. But a more flexible solver—one with a deep knowledge of procedures—can navigate his or her way through this procedural domain, using techniques other than ones that are overpracticed, to produce solutions that best match problem conditions or solving goals. I consider this kind of flexible knowledge to be both procedural and deep. Flexibility is not well explained or even accounted for in typical definitions of conceptual and procedural knowledge.

THE IMPLICATIONS OF RECONCEPTUALIZING PROCEDURAL KNOWLEDGE

Reconceptualizing procedural knowledge as described above has important implications for both research and practice. First and foremost, recognizing the existence of deep procedural knowledge suggests the need for research on what it is, how it develops, and what its relationship is to other types of desired mathematical knowledge. Broadening the definition of procedural knowledge could bring procedures back onto the research agenda of mathematics educators—including those on both "sides" of the math wars. Second, accompanying these new avenues for research is a need to broaden current ways of studying and assessing procedural knowledge. Methods for assessing students' procedural knowledge are somewhat impoverished at present, with procedural knowledge often measured simply by what a student can or cannot do. Research methods can instead focus on *how* students can and cannot do and on the character of the knowledge they have (including its depth), which supports their ability to perform procedures. And third, deep procedural knowledge should be considered an instructional goal at all levels of schooling. If so, additional research would be needed to develop and evaluate instructional interventions and curricula that might achieve this goal, as well as to determine the kinds of content knowledge for teaching that could support the development of deep procedural knowledge.

REFERENCES

Anderson, J. R. (1982). Acquisition of cognitive skill. *Psychological Review, 89,* 369–406.

Anderson, J. R. (1992). Automaticity and the ACT* theory. *American Journal of Psychology, 105,* 165–180.

Anderson, J. R., Fincham, J. M., & Douglass, S. (1997). The role of examples and rules in the acquisition of a cognitive skill. *Journal of Experimental Psychology: Learning, Memory, and Cognition, 23,* 932–945.

Ballheim, C. (1999, October). Readers respond to what's basic. *Mathematics Education Dialogues, 3,* 11.

Brown, J. S., & VanLehn, K. (1980). Repair theory: A generative theory of bugs in procedural skills. *Cognitive Science, 4,* 379–426.

Brown, S., Seidelmann, A., & Zimmermann, G. (n.d.). *In the trenches: Three teachers' perspectives on moving beyond the math wars.* Retrieved June 1, 2005, from the Mathematically Sane Web site: http://mathematicallysane.com/analysis/trenches.asp

Brownell, W. A. (1945). When is arithmetic meaningful? *Journal of Educational Research, 38,* 481-498.

Budd, K., Carson, E., Garelick, B., Klein, D., Milgram, R. J., Raimi, R. A., et al. (2005). *Ten myths about math education and why you shouldn't believe them.* Retrieved June 1, 2005, from the New York City HOLD Web site: http://www.nychold.com/myths-050504.html

Davis, R. B. (1983). Complex mathematical cognition. In H. P. Ginsburg (Ed.), *The development of mathematical thinking* (pp. 253–290). New York: Academic Press.

De Jong, T., & Ferguson-Hessler, M. (1996). Types and qualities of knowledge. *Educational Psychologist, 31,* 105–113.

Gelman, S. A., Star, J. R., & Flukes, J. (2002). Children's use of generics in inductive inferences. *Journal of Cognition and Development, 3,* 179–199.

Glaser, R. (1991). The maturing of the relationship between the science of learning and cognition and educational practice. *Learning and Instruction, 1,* 129–144.

Hiebert, J. (Ed.). (1986). *Conceptual and procedural knowledge: The case of mathematics.* Hillsdale, NJ: Lawrence Erlbaum.

Hiebert, J., & Lefevre, P. (1986). Conceptual and procedural knowledge in mathematics: An introductory analysis. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: The case of mathematics* (pp. 1–27). Hillsdale, NJ: Erlbaum.

Mathematically correct. (n.d.). Retrieved June 1, 2005, from the Mathematically Correct Web site: http://www.mathematicallycorrect.com/

Matz, M. (1980). Towards a computational theory of algebraic competence. *Journal of Mathematical Behavior, 3,* 93–166.

Medin, D. L. (1989). Concepts and conceptual structure. *American Psychologist, 44,* 1469–1481.

National Research Council. (2001). *Adding it up: Helping children learn mathematics.* J. Kilpatrick, J. Swafford, & B. Findell (Eds.). Washington, DC: National Academy Press.

Ohlsson, S., & Rees, E. (1991). The function of conceptual understanding in the learning of arithmetic procedures. *Cognition and Instruction, 8,* 103–179.

Pesek, D. D., & Kirschner, D. (2000). Interference of instrumental instruction in subsequent relational learning. *Journal for Research in Mathematics Education, 31,* 524–540.

Rittle-Johnson, B., Siegler, R. S., & Alibali, M. W. (2001). Developing conceptual understanding and procedural skill in mathematics: An iterative process. *Journal of Educational Psychology, 93,* 346–362.

Schoenfeld, A. H. (1979). Explicit heuristic training as a variable in problem-solving performance. *Journal for Research in Mathematics Education, 10,* 173–187.

Skemp, R. R. (1976). Relational understanding and instrumental understanding. *Mathematics Teaching, 77,* 20–26.

Sowder, J. T. (1998). What are the "math wars" in California all about? Reasons and perspectives. Retrieved June 1, 2005, from the Mathematically Sane Web site: http://mathematicallysane.com/analysis/mathwars.asp

Star, J. R. (2000). On the relationship between knowing and doing in procedural learning. In B. Fishman & S. O'Connor-Divelbiss (Eds.), *Proceedings of the Fourth International Conference of the Learning Sciences* (pp. 80–86). Mahwah, NJ: Erlbaum.

Star, J. R. (2002a). Re-conceptualizing procedural knowledge: The emergence of "intelligent" performances among equation solvers. In D. Mewborn, P. Sztajn, D. White, H. Wiegel, R. Bryant, & K. Nooney (Eds.), *Proceedings of the Twenty-Fourth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 999–1007). Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Star, J. R. (2002b). Re-conceptualizing procedural knowledge: Innovation and flexibility in equation solving (Doctoral dissertation, University of Michigan, 2001). *Dissertation Abstracts International, 62,* 3327A.

Star, J. R., & Seifert, C. (in press). The development of flexibility in equation solving. *Contemporary Educational Psychology.*

VanLehn, K. (1990). *Mind bugs: The origins of procedural misconceptions.* Cambridge, MA: MIT Press.

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